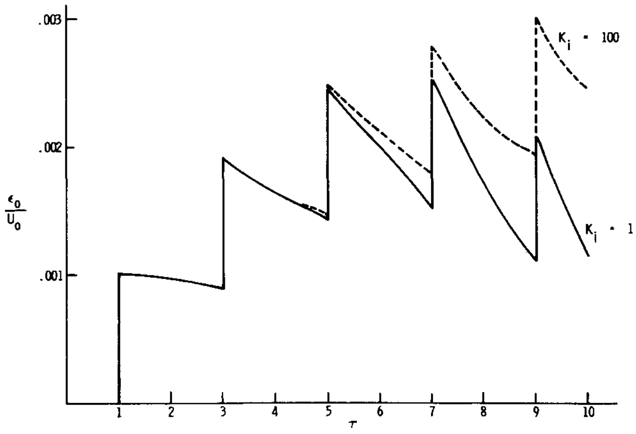

Fig. 2 Shell strains as a function of K_i for $M_o = 1000$, $M_i = 500$, $K_o = 5$, and $R = 40$.

where r_{wf} is the location of the wave front at the time for which the series is being evaluated. When Eq. (12) is subtracted from the expression for ε_r, a more rapidly convergent series results

$$\varepsilon_r - f(r) = U_o \Bigg\{ \left[\frac{R+1}{r_{wf}}\right]^{1/2} \frac{\tau}{2\pi M_o} + \sum_{n=1}^{\infty} \left[\frac{\phi_n(R+1)(d\phi_n/dr)}{\omega_n Q_n} \sin \omega_n \tau\right] - \sum_{m=1}^{\infty} \left[\frac{R+1}{r_{wf}}\right]^{1/2} \frac{2}{\pi M_o m} \sin(m\pi\tau/2) \cos\left(m\pi \frac{r-R-1}{2}\right) \Bigg\} \tag{14}$$

Thus ε_r is obtained as the sum of a discontinuous function known in closed form and a rapidly convergent infinite series. Equation (14) is not valid for limiting cases where either M_o or M_i is equal to zero because the asymptotic form in Eq. (11) for $\eta_n(d\phi_n/dr)$ changes. In these cases the procedure used above is still applicable, but a different discontinuous function must be chosen.

Numerical Example

Inspection of Eqs. (1) shows that the governing equations contain five parameters M_o, M_i, K_o, K_i and R. For a numerical example, values of four of the parameters were specified and the fifth, K_i, was allowed to vary. The example was chosen to demonstrate how variations of the inner shell stiffness influence the peak strain in the outer shell. The parameters were chosen as

$$R = 40, \quad M_o = 1000, \quad M_i = 500, \quad K_o = 5, \quad 0.1 < K_i < 1000$$

Figure 2 shows a plot of the peak outer and inner shell strains as a function of K_i. The curves show that for K_i less than 1, the inner shell gives negligible support to the outer shell. For K_i greater than 100, the inner shell is essentially rigid, which means that an increase of the inner shell stiffness has little effect on the amount of support for the outer shell. Figure 3 shows plots of radial strain in the center of the core layer for $K_i = 1$ and $K_i = 100$. These results were obtained by truncating the series in Eq. (14) at $n = 9$ and $m = 7$.

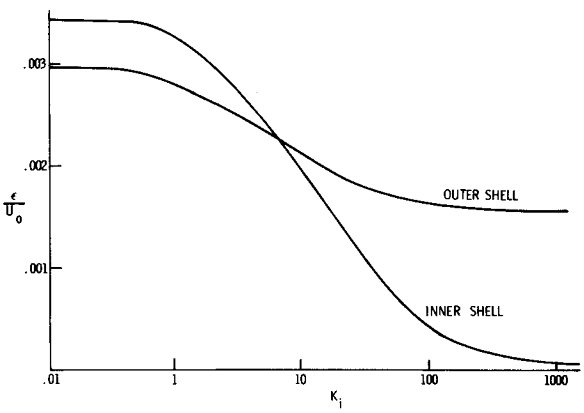

Fig. 3 Strains at the center of the core for $M_o = 1000$, $M_i = 500$, $K_o = 5$, and $R = 40$.

Summary and Conclusions

Modal solutions for the strains in a three-layered shell structure have been derived. These solutions are valid for all times, are rapidly convergent, and can easily be programed for evaluation by a computer. By setting $K_i = \infty$ or $K_i = M_i = 0$ the limiting cases of a rigid reflector or no internal shell can be treated. The solution for radial strain in the core contains a propagating discontinuity which, while useful, cannot be expected to give an exact detailed description of the wave front because of the assumption that the shell layers are infinitely rigid in the radial direction.

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A Note on Interacting Boundary Layers

H. E. Eickhoff*
Institut für Überschalltechnik, Technische Universität, Berlin, F.R. Germany

Nomenclature

$f_1, f_2, \bar{f}_2, f$ = dimensionless stream functions
L = length of moving surface
t = time
T = temperature
u = velocity in x-direction
u_s = shock velocity
U = freestream velocity
x, y = coordinates fixed to the leading edge (A)
$\bar{x}, \bar{y}$ = coordinates fixed to the shock (B) and the trailing edge (B), respectively
α = dimensionless length coordinate
$\eta, \bar{\eta}$ = dimensionless transverse coordinate
μ = dynamic viscosity
ρ = density
ν = kinematic viscosity

Subscript
w = at the wall

Introduction

The problem of two interacting boundary layers, where the equations are singular-parabolic first was considered by Lam and Crocco.[1] They numerically investigated the shock induced boundary layer on a semi-infinite flat plate for infinitely weak shock waves. Further numerical solutions of the boundary-layer equations have been obtained for finite shock strength.[2,3]

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* Assistant Professor.

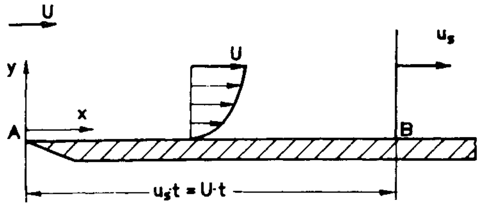

Fig. 1 Shock wave moving over a plate.

If the shock becomes infinitely strong, then both edges of the interaction region AB are singular points, Fig. 1. A similar flow situation where, at both edges of a region AB of interacting boundary layers, singularities occur, is the uniform flow past a finite flat surface continuously moving with negative velocity, Fig. 2.

In order to calculate the flowfields, it is necessary to specify two more boundary conditions at the edges A and B of the interaction regions, in addition to the usual wall and freestream conditions. These additional boundary conditions are the solutions of the boundary-layer equations at the edges and, as they are singular points, it is necessary to remove the singularities from the equations by suitable transformations. Up to now only the singularity at the leading edge A has been removed and the boundary-layer equations in the region of interaction were solved numerically by different methods[2,4] with some uncertainty with regard to the flow region near the singularity at B. The purpose of this Note is to study the latter range by appropriate transformations.

The Shock Induced Boundary Layer

With infinitely large shock strength, the flow velocity U behind the shock wave becomes equal to the shock velocity u_s, and influences from the boundary layer developing at the leading edge reach down to the shock boundary layer and the other way round. The mutual influence of these two boundary layers vanishes at the singular points A and B.

At the leading edge with

$$\eta = \int (\rho/\rho_w) dy \cdot (U \cdot \rho_w / x \cdot \mu_w)^{1/2}; \quad (f_1)_\eta = (u/U) \tag{1}$$

assuming $\mu \sim T$ for simplicity, the momentum equation with $x \to 0$ becomes[3]

$$(f_1)_{\eta\eta\eta} + \tfrac{1}{2} f_1 (f_1)_{\eta\eta} = 0 \tag{2}$$

where x and y are coordinates fixed to the leading edge. The boundary conditions are

$$\eta = 0 : (f_1)_\eta = 0, \quad f_1 = 0; \quad \eta \to \infty : (f_1)_\eta = 1 \tag{3}$$

Equation (2) with boundary conditions (3) applies to the uniform flow past a semi-infinite flat plate.

Accordingly, the singularity from the equation of the shock boundary layer can be removed. With $\bar{x}$, $\bar{y}$ as shock-fixed coordinates and

$$\bar{\eta} = \int (\rho/\rho_w) d\bar{y} \cdot (U \cdot \rho_w / \bar{x} \cdot \mu_w)^{1/2}; \quad (f_2)_{\bar{\eta}} = u/U \tag{4}$$

the momentum equation becomes with $\bar{x} \to 0$

$$(f_2)_{\bar{\eta}\bar{\eta}\bar{\eta}} + \tfrac{1}{2} f_2 (f_2)_{\bar{\eta}\bar{\eta}} = 0 \tag{5}$$

The boundary conditions are

$$\bar{\eta} = 0 : (f_2)_{\bar{\eta}} = 1, \quad f_2 = 0; \quad \bar{\eta} \to \infty : (f_2)_{\bar{\eta}} = 0 \tag{6}$$

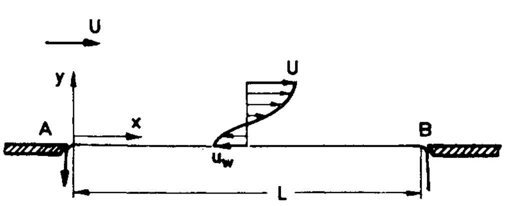

Fig. 2 Uniform flow past a finite flat plate with negative surface velocity.

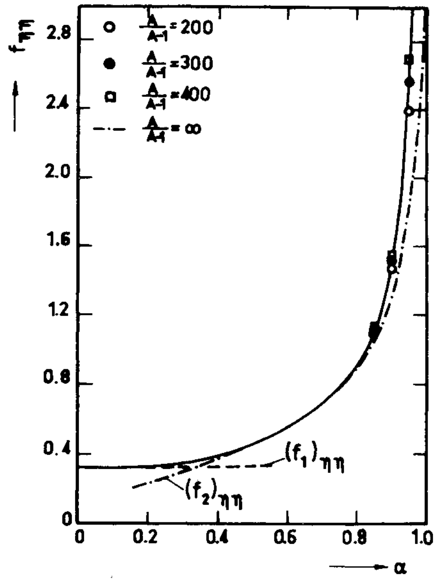

Fig. 3 Wall shear stress profile for the shock induced boundary layer —— after Ref. 2; - - - similarity solutions.

Equation (5) with boundary conditions (6) applies to the boundary-layer flow induced by an infinitely strong shock at a wall of infinite length. A different interpretation of practical interest is the boundary layer on a semi-infinite plane surface continuously moving from a slot into a fluid at rest, which numerically has been treated by Sakiadis.[5]

In Fig. 3, the self-similar solution of the shear stresses $(f_1)_{\eta\eta}$ and $(f_2)_{\bar{\eta}\bar{\eta}}$ at the wall are continued over their range of applicability into the region of interaction $0 < \alpha < 1$, with $\alpha = x/U \cdot t$, and compared to the numerical solution given by Felderman.[2]

The dimensionless wall shear stress at the leading edge is the flat plate relation $(f_1)_{\eta\eta} = 0.332$ and the dimensionless wall shear stress at the shock is $(f_2)_{\bar{\eta}\bar{\eta}} = 0.444$ which, transformed to the leading edge fixed coordinates, is

$$(f_2)_{\eta\eta} = 0.444(\alpha/1-\alpha)^{1/2}$$

In the range $0 < \alpha < 0.2$ and $0.4 < \alpha < 0.8$ the continued self-similar solutions agree with the numerical results and it can be seen from Fig. 3 that the real interaction region is $0.2 < \alpha < 0.4$. As the shear stress becomes infinite, with $\alpha \to 1$ presenting difficulties with numerical solution, the numerical procedure was performed for different large but finite values of the shear stress at $\alpha = 1$, which were taken from the self-similar solution of the shock boundary layer for finite shock strength $U/U_s < 1$. For values of α below 0.8 the influence of the magnitude of the boundary condition at $\alpha = 1$ on the numerical solution is vanishing. However, when $\alpha > 0.8$ the numerical solution with increasing shock strength, which is denoted by the ratio $A/(A-1)$, with $A = u_s/U$, does not converge to the self-similar solution which, with $\alpha \to 1$, becomes exact.

The Boundary Layer on a Flat Plate with Negative Surface Velocity

As shown in Ref. 4, there exists a self-similar solution at the leading edge which corresponds to the uniform flow past a semi-infinite flat surface with negative velocity $-u_w$

$$(f_1)_{\eta\eta\eta} + \tfrac{1}{2} f_1 (f_1)_{\eta\eta\eta} = 0 \tag{7}$$

with

$$\eta = y(U/\nu \cdot x)^{1/2} \quad \text{and} \quad (f_1)_\eta = u/U \tag{8}$$

The coordinates x and y are fixed to the leading edge. The boundary conditions are

$$\eta = 0 : (f_1)_\eta = (-u_w/U), \quad f_1 = 0; \quad \eta \to \infty : (f_1)_\eta = 1 \tag{9}$$

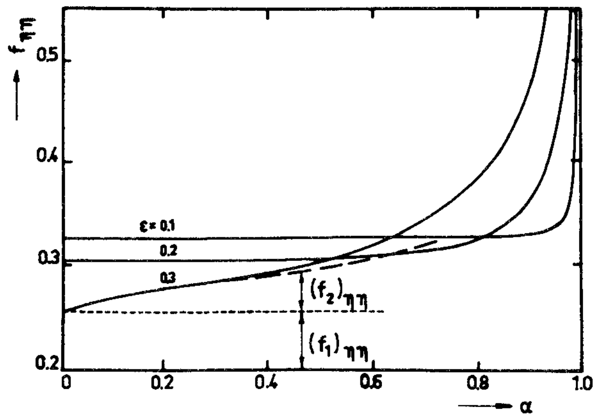

Fig. 4 Wall shear stress profiles for the continuously moving surface for various $\varepsilon = u_w/U$ after Ref. 4; ---similarity solution.

The transformation eliminating the singularity at the trailing edge is

$$\bar{\eta} = \bar{y}(u_w/v \cdot \bar{x})^{1/2}, \quad (\bar{f}_2)_{\bar{\eta}} = (u/-u_w) \tag{10}$$

where the coordinates $\bar{x}$, $\bar{y}$ are fixed to the trailing edge. With $\bar{x} \to 0$ the transformed momentum equation becomes

$$(\bar{f}_2)_{\bar{\eta}\bar{\eta}\bar{\eta}} + \tfrac{1}{2}\bar{f}_2(\bar{f}_2)_{\bar{\eta}\bar{\eta}} = 0 \tag{11}$$

The boundary conditions are at the wall

$$(\bar{f}_2)_{\bar{\eta}} = 1, \quad \bar{f}_2 = 0 \tag{12.1}$$

and at the outer edge of the boundary layer

$$(\bar{f}_2)_{\bar{\eta}} = 0 \tag{12.2}$$

Equation (11) with boundary conditions (12.1) and (12.2) applies to the region of reverse flow which, with $\bar{x} \to 0$, becomes independent of the forward flow.

In Fig. 4, the dimensionless wall shear stress as given in Ref. 4 is shown as function of $\alpha = x/L$ for different values of $\varepsilon = u_w/U$, where L is the length of the moving surface. Using transformation (8), the problem numerically was solved in the interaction region $0 < \alpha < 1$. Since this transformation removes the singularity at the leading edge only, the second singularity at the trailing edge $\alpha = 1$ made problems for the numerical solution affecting its accuracy near $\alpha = 1$, as pointed out in Ref. 4.

The asymptotic value of the wall shear stress at the trailing edge can now be taken from the solution of Eq. (11) which is $(\bar{f}_2)_{\bar{\eta}\bar{\eta}} = 0.444$, as in the foregoing example of the shock induced boundary layer. Transformed to the coordinates being fixed to the leading edge, the asymptotic value of the wall shear stress shown in Fig. 4 with $\alpha \to 1$ is

$$f_{\eta\eta} = (f_2)_{\eta\eta} = 0{,}444\,\varepsilon^2 \cdot (\alpha/1-\alpha)^{1/2}$$

It is interesting to note that for stronger reverse flow ($\varepsilon = 0.3$) although the problem is nonlinear, linear superposition of the wall shear stresses, $(f_1)_{\eta\eta}$ and $(f_2)_{\eta\eta}$, near the leading edge, yields agreement with the numerical solution, as can be seen from Fig. 4.

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Taylor-Görtler Instability of a Boundary Layer with Suction or Blowing

Ryōji Kobayashi*
Tohoku University, Sendai, Japan

Introduction

The purpose of this Note is to consider theoretically how the suction or blowing from a slightly concave permeable wall will affect the instability of the incompressible two-dimensional laminar boundary layer to the onset of longitudinal vortices. As is well known, the critical Reynolds number for Tollmien-Schlichting waves depends strongly on the basic profile of the streamwise velocity component in the laminar boundary layer, while the critical Görtler parameter for the longitudinal vortices varies insensibly with a large change of the streamwise velocity profile.[1,2] We are here interested in a role of the normal velocity component in the Taylor-Görtler instability. In particular, suction or blowing from a permeable wall induces a notable normal velocity component in the boundary layer.

Analysis

A freestream with a uniform velocity U_∞ is directed along the axis x, where x is the arc length of the basic concave streamlines, y is the distance measured perpendicular to the wall and z normal to the two axes, (u, v, w) are the corresponding components of the velocity vector. The radius R of the curvature on the wall remains constant in the x-direction and is far larger than the momentum thickness θ of the boundary layer. The normal velocity profile v_0 shows in general a wide variety in dependence on a manner of suction or blowing. We put the suction distribution $v_0 = -C(vU_\infty/x)^{1/2}/2$ along the surface, where v denotes the kinematic viscosity of the fluid. In this case the boundary-layer flow has a similar solution.[3] We shall now suppose that the basic laminar boundary-layer flow is slightly perturbed with the type of longitudinal vortices, which may be expressed as

$$u = u_0(x, y) + u_1(y)\cos\alpha z \exp\left(\textstyle\int \beta\, dx\right) \tag{1a}$$

$$v = v_0(x, y) + v_1(y)\cos\alpha z \exp\left(\textstyle\int \beta\, dx\right) \tag{1b}$$

$$w = w_1(y)\sin\alpha z \exp\left(\textstyle\int \beta\, dx\right) \tag{1c}$$

with the wavenumber α and a measure β of the rate of growth of the disturbances. We have finally a system of perturbation equations governing the present linear instability problem in the neutral state ($\beta = 0$) as follows:

$$(D^2 - \bar{v}_0 D + D\bar{v}_0 - \sigma^2)\bar{u}_1 = (D\bar{u}_0)\bar{v}_1 \tag{2}$$

$$(D^2 - \bar{v}_0 D - D\bar{v}_0 - \sigma^2)(D^2 - \sigma^2)\bar{v}_1 = -2\sigma^2 G^2 \bar{u}_0 \bar{u}_1 \tag{3}$$

where $\bar{u}_0 = u_0/U_\infty$, $\bar{v}_0 = v_0\theta/v$, $\bar{u}_1 = u_1/U_\infty$, $\bar{v}_1 = v_1\theta/v$, $\sigma = \alpha\theta$, $G = (U_\infty\theta/v)(\theta/R)^{1/2}$, $\eta = y/\theta$ and $D = d/d\eta$. G is called Görtler parameter. Equations (2) and (3) for $\bar{v}_0 = 0$ coincide with the classical equations.[1,2]

We assume no slip condition at the permeable surface ($\eta = 0$). This assumption may be approximately valid when holes or slits of the permeable wall are fine enough. The boundary conditions then become $\bar{u}_1 = \bar{v}_1 = D\bar{v}_1 = 0$ in consideration of the equation of continuity. It may be reasonable to take the other boundary condition at a point ($\eta \to \infty$) far enough from the wall, because the longitudinal vortices will extend over the outer edge of the boundary layer. The boundary condition under which all perturbations vanish at the infinity ($\eta \to \infty$) leads again to $\bar{u}_1 = \bar{v}_1 = D\bar{v}_1 = 0$. It is, however, necessary in numerical integrations to give a finite value (η_m) instead of the infinity. The value η_m should be taken greater as

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* Professor, Institute of High Speed Mechanics.